

# Compact Aspherical Manifolds Whose Fundamental Groups Have Nontrivial Center

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## 1 Introduction

Aspherical manifolds are manifolds with contractible universal cover. Such manifolds have the homotopy type (and the homeomorphism type, in many cases) determined by their fundamental groups and play an important role in mathematics. Many beautiful examples of aspherical manifolds are of the form  $K \backslash G / \Gamma$ , where  $G$  is a connected Lie group,  $K$  its maximal compact subgroup, and  $\Gamma$  a torsion free uniform lattice in  $G$ .

Much of what we know and conjecture about aspherical manifolds is based on analogy to the Lie theoretic setting. For instance, a theorem of Borel asserts that, if the fundamental group of a compact aspherical manifold is centerless, then the only compact connected group that can act effectively on the manifold is the trivial group [4]. Moreover, the finite groups that act effectively must act as outer automorphisms of the fundamental group.

When  $G$  is semisimple, the quotient  $K \backslash G$  is nonpositively curved with no Euclidean factors, and the lattice  $\Gamma$  is centerless. The more general situation leads to generalizations of Seifert fibered spaces and were analyzed in a number of papers by Conner and Raymond [13, 14, 16, 30]. In particular, Conner and Raymond proved in [13] that only toral groups, among the connected Lie groups, can act on closed aspherical manifolds, and the centers of their fundamental groups must contain an injective image of fundamental group of the torus. They further raised the question [13, page 229] whether a converse to Borel's theorem holds<sup>1</sup>: If the fundamental group of an aspherical manifold has nontrivial center,

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\*Research was partially supported by an NSF grant

<sup>†</sup>Research was partially supported by an NSF grant

<sup>‡</sup>Research was supported by Hong Kong Research Grant Council

<sup>1</sup>Borel's analysis actually shows that for a torus action on an aspherical manifold, the orbit of any point is central in the fundamental group, and the inclusion map is an injection on fundamental group.

then the manifold has a circle action, such that the orbit circle is a nontrivial central element of the fundamental group. The conjecture was stated again recently in [29]. This conjecture is true in dimension 3 by the work of [9, 26].

In the decades since this speculation arose, many of the conjectures about aspherical manifolds from that period were disproved. Essentially, the only one that remains is the Borel conjecture that asserts a form of topological rigidity for aspherical manifolds, see [2, 22] for some of the more recent positive results on these problems.

In a sense, the conjecture about the center could be thought of as a very strong form of the Borel conjecture. In fact, any central element of the fundamental group of an aspherical manifold can be realized by a map of the circle into the space of self homotopy equivalences. What we are asking here is for this circle family of self homotopy equivalences to be a compact 1-parameter group. It turns out we asked too much, and the conjecture is false.

The counterexamples of most of the old conjectures stem from essentially two different constructions. The first is the “reflection group trick” of Mike Davis [17] that gave rise to the first aspherical manifolds whose universal covers are not Euclidean spaces. Such aspherical manifolds have no differentiable structure and are not triangulable. Moreover, their fundamental groups are not residually finite, and may even have unsolvable word problem. See [18] for a survey and references.

The second construction is Gromov’s idea of hyperbolization [27], developed in [10, 19, 20, 31]. It implies that aspherical manifolds exist in abundance. For instance, any compact PL manifold is the image of an aspherical manifold by a degree one tangential map, and any cobordism class can be represented by an aspherical manifold.

In both constructions, the fundamental groups of the aspherical manifolds are centerless. Interestingly, Lee and Raymond [30] showed that if the fundamental group of an aspherical manifold has nontrivial center, or more generally contains a nontrivial abelian normal subgroup, then the universal cover is homeomorphic to an Euclidean space. This is rather uncommon in the setting of Davis constructions.

In this note, motivated by the work of Conner and Raymond, we will give some counterexamples for which the fundamental group has nontrivial center. Ultimately the result is a combination of surgery method [6, 7], a construction of non-arithmetic hyperbolic manifolds [28], a version of the hyperbolization [20], a construction for the Nielsen realization problem [1], the characterization of homologically injective toral actions [14], and homology manifolds [5] (as the theory in [14] does not guarantee that the reduction to a Nielsen type problem will necessarily be on a manifold). As a technical aside, we point out, following a suggestion of Jim Davis, how to use [11] to simplify [1]. To get examples in all sufficiently large dimensions, it is necessary to develop a bit of a theory of topological  $G$ -surgery up to “pseudoequivalence”. See [32] for an influential early treatment in the smooth category. There are a number of interesting complications in the topological category, and we provide just enough for our applications.

**Theorem.** *At any dimension  $\geq 6$ , there are aspherical manifolds with fundamental groups having  $\mathbb{Z}$  as the centers, yet do not possess nontrivial topological circle actions.*

The examples are mapping tori of homeomorphisms  $h: M \rightarrow M$ , that are not homotopic to involutions despite the fact that  $h^2 = h \circ h$  are homotopic to the identity. To get examples at all dimensions, we may take products of these with some hyperbolic

manifolds. The manifolds  $M$  and the homotopy classes of  $h$  are constructed in [1], although the fact that the homotopy classes contain homeomorphisms was not noted in that paper. The infinite cyclic center is generated by the square of the stable letter of the HNN-extension representing the fundamental group of the mapping torus.

We divide the proof into several small steps, some of which are, at least implicitly, quite old, but included for the sake of readability.

## 2 Construction

**Step 1.** Equivariant non-rigidity of  $\mathbb{Z}_2$ -tori.

Borel conjectured that aspherical manifolds are topologically rigid. The equivariant version of this conjecture asserts that aspherical manifolds with suitable finite or compact group actions are also rigid? However, the equivariant Borel conjecture is false. The simplest counterexamples are involutions on tori, which we will use in our construction. Very recently, Connolly, Davis and Kahn [12] gave a very detailed and complete analysis of the equivariant structure set of such involutions.

For closed manifolds, the surgery exact sequence

$$\rightarrow L_{n+1}(\pi, \omega) \rightarrow S(M) \rightarrow H_n(M, \mathbb{L}) \rightarrow L_n(\pi, \omega)$$

is an exact sequence of abelian groups that computes the  $s$ -cobordism class  $S(M)$  of homology manifolds simple homotopy equivalent to  $M$ . The  $L$ -groups  $L_n$  are defined in [38] and depend on the dimension  $n$ , the fundamental group  $\pi$ , and orientation character  $\omega$ . The homology  $H_n$  is the generalized homology theory associated to the (simply connected)  $L$ -spectrum  $\mathbb{L}$ . For noncompact manifolds, the exact sequence (where the homology is the ordinary one) computes the structures that have compact support, i.e., for which the homotopy equivalences are homeomorphisms (or CE-maps) outside of some compact subsets. If the  $L$ -group is replaced by a relative  $L$ -group that takes into account the fundamental group at infinity, and the homology is the locally finite homology, then the exact sequence computes the proper structures.

We will not worry about decorations in the surgery exact sequence, since all the discrete groups we are considering will have vanishing Whitehead groups, so that there is no difference between different decorations.

Let  $T$  be the torus  $T^n$  with standard involution, such that the fixed set has dimension either 0 or 1. In the surgery exact sequence, we take  $M$  to be (a compactification of) the complement of the fixed point of  $T$ . Thus  $\pi$  is the orbifold fundamental group of the involution, and is a semidirect product  $\mathbb{Z}^n \rtimes \mathbb{Z}_2$  according to the representation of  $\mathbb{Z}_2$  on  $H_1(T)$ . This representation is either the multiplication by  $-1$  (when fixed set is discrete) or has a one dimensional trivial summand (when fixed set has dimension 1).

The semidirect product  $\pi = \mathbb{Z}^n \rtimes \mathbb{Z}_2$  has  $\mathbb{Z}^{d+1} \rtimes \mathbb{Z}_2 \cong \mathbb{Z}^d \times (\mathbb{Z}_2 * \mathbb{Z}_2)$  as a direct summand, where  $d$  is the dimension of the fixed set, and  $\mathbb{Z}_2$  acts trivially on the  $\mathbb{Z}^d$  part of  $\mathbb{Z}^{d+1}$ . The surgery obstruction group of the direct summand is given as follows.

**Theorem** (Connolly and Davis [11]). *The surgery obstruction group  $L_r(\mathbb{Z}_2 * \mathbb{Z}_2, \omega \oplus \omega)$  is infinitely generated if and only if either  $\omega$  is trivial and  $r = 2$  or  $3 \bmod 4$ , or  $\omega$  nontrivial*

and  $r = 0$  or  $1 \bmod 4$ . Moreover, crossing with a circle produces infinitely generated subgroups in  $L_{r+1}(\mathbb{Z} \times (\mathbb{Z}_2 * \mathbb{Z}_2), \omega \oplus \omega)$  in the same cases.

The case of  $\omega$  trivial and  $r = 2 \bmod 4$  is due to Cappell [6] and is expressed there as the geometric statement that there are infinitely many manifolds that are homotopy equivalent to  $\mathbb{R}P^{4k+1} \# \mathbb{R}P^{4k+1}$ , but are not the connected sums of two manifolds homotopy equivalent to  $\mathbb{R}P^{4k+1}$ .

The actual result in [11] is a calculation of the UNil groups of Cappell [7] that describe the failure of reduced  $L$ -theory to be additive for free products (or more generally the Mayer-Vietoris sequences associated to group actions on trees). They subsequently analyzed the complete obstruction to this additivity problem. Actually, the groups in the theorem are infinitely generated exactly when the UNil groups are nontrivial. In case the fixed set is discrete, Connolly, Davis and Kahn [12] further showed that  $S(M)$  is a sum of such UNil groups. Moreover, they also showed that  $S(M)$  is the same as the isovariant structure set  $S^{\text{iso}}(T)$  of exotic involutions on the torus that are isovariantly homotopic to the standard involution. Modified by a formula of Shaneson [36], we have the similar description when the fixed set has dimension 1.

For our purpose, all we need is that a nontrivial element of  $L_{n+1}(\mathbb{Z}^d \times (\mathbb{Z}_2 * \mathbb{Z}_2), \omega \oplus \omega)$  gives, via split injections  $\mathbb{Z}^d \times (\mathbb{Z}_2 * \mathbb{Z}_2) \rightarrow \pi$ , an involution on the torus that is homotopic to the standard involution but not isovariantly homeomorphic to it. We denote the torus with this exotic involution by  $T'$ . It is important to note that the exotic involution remains PL. By proper surgery, the exotic involution do not become standard even when the fixed sets are deleted. Thus, there is no need for stratified surgery techniques [41] for this part of the analysis.

When is  $L_{n+1}(\mathbb{Z}^d \times (\mathbb{Z}_2 * \mathbb{Z}_2), \omega \oplus \omega)$  nontrivial, so that exotic involutions exist? If  $d = 0$  and  $n$  is even, then  $\omega$  is trivial, and  $L_{n+1}$  is nontrivial only when  $n + 1 = 2$  or  $3 \bmod 4$ . Therefore  $n = 2 \bmod 4$ . If  $d = 0$  and  $n$  is odd, then  $\omega$  is nontrivial, and  $L_{n+1}$  is nontrivial only when  $n + 1 = 0$  or  $1 \bmod 4$ . Therefore  $n = 3 \bmod 4$ .

We conclude that there are exotic involutions on  $T^n$  with discrete fixed set if and only if  $n = 2$  or  $3 \bmod 4$ . Similarly, there are exotic involutions on  $T^n$  with 1-dimensional fixed set if and only if  $n = 0$  or  $3 \bmod 4$ . Relying on [6], Block and Weinberger [1] made use of these elements, with a simpler analysis in the  $d = 0$  case.

## Step 2. Counterexample to Nielsen realization problem.

The torus  $T$  with the standard involution is easily the boundary of a PL aspherical manifold  $W$  with involution, such that the inclusion  $\pi_1 T \rightarrow \pi_1 W$  is injective. We will show that the same is true for the torus  $T'$  with exotic involution constructed in the first step.

The construction of  $T'$  shows that there is a normal cobordism  $Z$  between the standard involution  $T$  and the exotic involution  $T'$ . Moreover, the cobordism is trivial on the fixed part, the involution action is finite and PL, and it can be arranged so that the orbifold fundamental group of  $Z$  is also  $\pi$ . Now we may apply the relative version [20] of Gromov hyperbolization [10, 19], which starts from any simplicial complex and combines “simplices of non-positively curved manifolds” to build a polyhedral space that closely resembles the polyhedron [18, 19]. So we hyperbolize the quotient  $Z/\mathbb{Z}_2$  relative to the two ends, and get

an aspherical  $\mathbb{Z}_2$ -cobordism  $Z'$  between  $T$  and  $T'$ . Then  $V = Z' \cup_T W$  is a PL aspherical manifold with involution, such that  $\partial V = T'$ , and the inclusion  $\pi_1 T' \rightarrow \pi_1 V$  is injective.

Although  $\partial W = T$  and  $\partial V = T'$  are not equivariantly homeomorphic, the involutions are homotopic. Any one such homotopy equivalence is homotopic to a homeomorphism. Let  $M$  be obtained by glueing  $W$  and  $V$  along this homeomorphism. The construction is somewhat reminiscent of the construction in [28] of non-arithmetic lattices by glueing hyperbolic manifolds with totally geodesic boundary together via isometries of their respective boundaries.

Consider  $M$  as  $W \cup_T T^n \times [0, 1] \cup_{T'} V$ . We have the standard involution on  $T^n \times 0$  and the exotic involution on  $T^n \times 1$ . Since homotopic homeomorphisms of the torus are always pseudoisotopic (see [38], for example), the involutions on both ends of the ribbon  $T^n \times [0, 1]$  can be extended to a (noninvolutive) homeomorphism of the interior. This gives a homeomorphism  $h: M \rightarrow M$  that satisfies  $h^2 = id$  on  $W$  and  $V$  but is not involutive on  $M$ .

Since  $h$  is involutive on  $W$  and  $V$ , we have  $h^2 \simeq id$  on  $M$ . However,  $h_*: \pi_1 M \rightarrow \pi_1 M$  is not a conjugation. In fact, it is shown in [1] that no manifold homotopy equivalent to  $M$  has an involution inducing the same outer automorphism as  $h_*$  on the fundamental group. In particular, the outer automorphism  $h_*$  on the fundamental group of  $M$  gives a counterexample to the generalized Nielsen realization problem.

Our desired aspherical manifold will be the mapping torus  $T(h)$ . To get arbitrary dimensions, it will be necessary to take the product of this with a closed hyperbolic manifold.

Note that if the exotic involution is concordant to the standard one, then the extension to the interior of the ribbon can be kept involutive, so that  $T(h)$  would have a circle action whose orbits would go through the section  $T^n$  (generically) twice. Our aim is to show that  $T(h)$  as constructed does not have any circle actions, nor does its product with any hyperbolic manifold.

### Step 3. Nielson for homology manifolds.

For our current application, we need to extend the result of [1] to homology manifolds homotopy equivalent to  $M$ . Moreover, we need to argue that the result remains true after crossing with a closed hyperbolic manifold.

The argument in [1] used surgery theory for topological manifolds. According to [5], for homology manifolds, the same arguments apply unchanged except for perhaps an extra  $\mathbb{Z}$  in the normal invariants. In fact, this extra  $\mathbb{Z}$  does not appear here, because our homology manifold has a Poincaré embedded codimension 1 torus. Given the fact that surgery theory holds for homology manifolds, there is a codimension one locally flat torus is a homology manifold homotopy equivalent to this one. Therefore, by an argument similar to one in [3], the homology manifold must automatically be resolvable, which disposes of this final  $\mathbb{Z}$ .

To obtain the fact that this remains true after crossing with a closed hyperbolic manifold, we need to apply the theory of surgery obstruction to pseudoequivalence. The basic problem is that ordinary equivariant or stratified surgery requires more information about the underlying stratified homotopy theory of the putative group action than Smith theory provides. Indeed, the fixed set of the actions that can occur need not even be ANRs, and

have fairly uncontrolled homotopy theory away from 2. In the situation of  $T(h)$ , the fixed set is low enough dimensional that the local homotopy theory forces the fixed set to be a manifold, and then more standard tools can be applied. However, as we observe below, some part of the total surgery obstruction for  $T(h)$  does apply a priori to our situation even after crossing with a positive dimensional manifold because this raises the dimension of the fixed set.

**Definition.** A pseudoequivalence  $f: X \rightarrow Y$  between  $G$ -spaces is an equivariant map which is a homotopy equivalence upon ignoring the group action.

If  $X$  and  $Y$  are finite dimensional, then Smith theory tells us that  $f$  induces some sorts of homological isomorphisms between fixed sets of subgroups of prime power order. The condition itself is equivalent to the statement that the map  $X \times EG \rightarrow Y \times EG$  is an equivariant homotopy equivalence.

In our situation, we have a  $Y$  that is a stratified Poincaré object, and the manifold solution of the Nielsen realization problem would be pseudoequivalent to  $Y$ . In [35], pseudoequivalence invariance of (higher) signature operators is studied. Unfortunately, the analysis there is not adequate for our situation that uses an obstruction at the prime 2. We shall not use a priori absolute invariants, but have to use relative invariants to capture our obstruction.

**Proposition.** Suppose that  $f: N \rightarrow Y$  is a degree one  $G$ -map from a manifold to an equivariant Poincaré complex that is a pseudoequivalence on the singular sets of these spaces, and that after crossing with  $EG$ , this map is covered by surgery bundle data. Then it is possible to define a surgery obstruction associated to  $f$  that lies in  $L_d^p(\pi_1((Y \times EG)/G))$ .

*Proof.* We observe that the usual definition of the surgery obstruction actually applies to the chain complex of the mapping cylinder of  $f$ , once one verifies that it is chain equivalent to a finite projective chain complex. This is easy, since the map is  $G$ -equivalence after crossing with any free  $G$ -space, so one can cross with a highly connected free  $G$ -manifold, use the free structure on that map, and then truncate at  $\dim Y$  ( $= \dim M = d$ ).

This truncated chain complex is projective as shown by Wall [37]. In general, for nonfree actions, one cannot improve this to a free chain complex as examples of [34] or [39] show.  $\square$

Since the surgery obstruction that we have obtained lies in  $L^p$ , we cannot always complete surgery when this obstruction vanishes. One can only complete the surgery after crossing with  $\mathbb{R}$ . We hope to return to the geometry of surgery up to pseudoequivalence in the topological category in a future paper. For our current purpose, the previous proposition defining the obstruction suffices.

In light of the proposition, if a stratified Poincaré complex is pseudoequivalent to a  $G$ -manifold, one sees that the top pure stratum, thought of as a pair, has a degree one normal map. The surgery obstruction of this vanishes in  $L_d^p(\pi_1((Y \times EG)/G), \pi_1((\Sigma \times EG)/G))$  ( $\Sigma$  denoting the singular set of  $Y$ ), because one obtains a homotopy equivalence after glueing in the neighborhood of  $\Sigma(N)$ .

In our situation, the UNil obstruction survives the removal of the singular set. This is seen by considering the boundary map in the exact sequence of an amalgamated free product that calculates  $\pi_1 M$  and then mapping further to the pair  $(\mathbb{Z}_2 * \mathbb{Z}_2, \mathbb{Z}_2 \sqcup \mathbb{Z}_2)$ .

Now, if we cross with a hyperbolic manifold, then the Novikov conjecture with coefficients in the ring  $\mathbb{Z}[\pi_1 M \rtimes \mathbb{Z}_2]$  implies that our UNil obstruction still survives. The reason is that the obstruction survives upon crossing with  $\mathbb{R}^n$ , as an element in the bounded  $L$ -theory over  $\mathbb{R}^n$  (see [23]). Thus we can take the transfer to the universal cover  $\mathbb{R}^n$  of the hyperbolic manifold, followed by the inverse of the exponential map. Actually, this application of bounded  $L$ -theory is behind the descent proofs of the Novikov conjecture with coefficients. See [8, 24, 25], for examples.

**Step 4.** Uniqueness of HNN-extension.

Since  $M$  is aspherical, its fundamental group is torsion free. As observed in [1], the fundamental group is also centerless. This is because it is constructed via an amalgamated free product along a subgroup containing no central elements. Indeed, the fundamental group of a hyperbolized cobordism generally contains  $\mathbb{Z} * \pi_1 \partial$ .

The fundamental group of the mapping torus  $T(h)$  is the HNN-extension

$$\pi_1(T(h)) = \pi_1 M *_{h_*} = \pi_1 M \rtimes_{h_*} \langle t \rangle, \quad tat^{-1} = h_*(a).$$

By  $h^2 \simeq id$ , we have  $h_*^2 = id$ . The HNN extension is unique up to conjugation.

**Lemma.** *Suppose  $G$  is a torsion free and centerless group. Suppose  $\alpha$  is an automorphism of  $G$  of finite prime order  $p$ , and  $\alpha$  is not a conjugation (i.e.,  $\alpha$  is a nontrivial outer automorphism). Suppose the HNN-extension  $G \rtimes_{\alpha} \langle t \rangle$  can be expressed as another HNN-extension  $H \rtimes_{\beta} \langle s \rangle$ , where  $\beta$  still has finite order. Then the two HNN-extensions are conjugate.*

*Proof.* Denote by  $Z(\pi)$  the center of group  $\pi$ . Let  $at^k \in Z(G \rtimes_{\alpha} \langle t \rangle)$ ,  $a \in G$ . We have  $at^k \cdot b = b \cdot at^k$  for any  $b \in G$ . This means that  $\alpha^k(b) = a^{-1}ba$ . Since  $\alpha$  is not a conjugation and has prime order  $p$ , we see that  $p$  divides  $k$ . Then  $\alpha^k = id$ , and  $a \in Z(G)$ . Since  $G$  is centerless, we get  $a = 1$  and conclude that the center of  $G \rtimes_{\alpha} \langle t \rangle$  is infinite cyclic and generated by  $t^p$ .

By the assumption, we have  $\beta^i = id$  for some natural number  $i$ . This means that  $s^i$  is central. Let  $s = at^j$ ,  $a \in G$ . Then  $s^j = bt^{ij}$  for some  $b \in G$ . Since  $s^j$  is central, we have  $b = 1$  and  $ij$  divisible by  $p$ .

If  $p$  divides  $j$ , then  $t^j$  is central, and  $s^i = a^{it^{ij}}$ . Thus  $a^i = b = 1$ , and by  $G$  torsionless, we get  $a = 1$ . So  $s = t^j$  is already central. Now let  $t = xs^l = xt^{jl}$ ,  $x \in H$ . We have  $x = t^{1-jl}$  and  $s = t^j$ . This implies that  $x^j = s^{1-jl} = t^{j(1-jl)}$  is an element of  $H$  as well as  $\langle s \rangle$ . Since the two subgroups have trivial intersection, we get  $t^{j(1-jl)} = 1$ . Since  $p$  divides  $j$ , we must have  $j = 0$ . Thus  $s = 1$ , a contradiction.

So  $p$  cannot divide  $j$  and must divide  $i$  instead. Let  $i = pl$ . Then  $s^i = (at^j)^{pl} = (a\alpha^j(a)\alpha^{2j}(a) \cdots \alpha^{(p-1)j}(a))^l t^{ij}$ . Thus

$$(a\alpha^j(a)\alpha^{2j}(a) \cdots \alpha^{(p-1)j}(a))^l = b = 1,$$

and by  $G$  torsionless, we get  $a\alpha^j(a)\alpha^{2j}(a) \cdots \alpha^{(p-1)j}(a) = 1$ . This implies that  $s^p = a\alpha^j(a)\alpha^{2j}(a) \cdots \alpha^{(p-1)j}(a)t^{pj} = t^{pj}$  is already central. Now let  $t = xs^l$ ,  $x \in H$ . Then  $t^p = ys^{pl} = yt^{pl}$  for some  $y \in H$ . We have  $y = t^{p(1-lj)} \in H$  and  $y^j = t^{pj(1-lj)} = s^{p(1-lj)} \in \langle s \rangle$ .

Since the intersection of  $H$  and  $\langle s \rangle$  is the trivial group, we get  $y^j = t^{pj(1-lj)} = 1$ , so that  $lj = 1$  and  $y = 1$ . This tells us

$$s = at, \quad a \in G; \quad t = xs, \quad x \in H.$$

Thus  $ax = 1$  and

$$G \rtimes_{\alpha} \langle t \rangle = H \rtimes_{\beta} \langle at \rangle = H \rtimes_{\beta} \langle x^{-1}t \rangle = H \rtimes_{x\beta x^{-1}} \langle t \rangle.$$

This implies that  $G = H$  and  $\alpha = x\beta x^{-1}$ .  $\square$

A geometric way to think about the lemma is to consider the HNN extensions in the lemma as if they were circle actions. Compare Step 5 below. On the two fold cover of  $T(h)$  there is clearly only one possible decomposition: The orbit would have lie in the center, and the stable letter would have a power lying in the center, but torsion freeness would make the stable letter central. Now, every circle action lifts to a finite cover (perhaps at the cost of changing speed), so one can apply the uniqueness on the cover to obtain uniqueness in the quotient.

At every dimension  $\geq 2$ , we can find closed hyperbolic manifolds with centerless fundamental group. By Preissman's Theorem [33] (or [21, page 260]), the fundamental groups of closed hyperbolic manifolds are always torsionless. Taking a product with such a closed hyperbolic manifold does not change the argument above.

**Step 5.**  $T(h)$  has no nontrivial circle action.

Suppose  $T(h)$  admits a nontrivial circle action. Since  $T(h)$  is an aspherical manifold, according to Borel the orbit of any point always defines a nontrivial element of the center of the fundamental group [15]. By Step 4, the center is infinite cyclic. Therefore the action is injective on homology.

By [14], homologically injective  $S^1$ -actions on any space always come from a balanced product construction

$$T(h) \cong (Y \times [0, 1]) / \sim$$

where  $\sim$  glues  $Y \times 1$  to  $Y \times 0$  via an automorphism  $\phi: Y \rightarrow Y$  of certain finite order  $n$ .

Since  $T(h)$  is an aspherical manifold,  $Y$  is necessarily an aspherical homology manifold. The balanced product gives another HNN decomposition  $\pi_1(T(h)) = \pi_1 Y \rtimes \langle s \rangle$ , such that the conjugation by  $s$  on  $\pi_1 Y$  is  $\phi_*$ . By Step 4, the two HNN decompositions differ by a conjugation. This implies that  $M \times \mathbb{R}$  and  $Y \times \mathbb{R}$  are the covers with respect to the same subgroup of the fundamental group, so that  $M$  and  $Y$  are homotopy equivalent. Moreover, the uniqueness of the HNN decompositions up to conjugation also shows that the elements  $h_* \in \text{Out}(\pi_1 M)$  and  $\phi_* \in \text{Out}(\pi_1 Y)$  are the same.

Borel showed that a finite group of homeomorphisms is represented faithfully in  $\text{Out}(\pi_1 Y)$ . Although the fact is proved in [15] for manifolds, the proof is homological in nature and applies, with no change, to homology manifolds. Applying Borel's result to the automorphism  $\phi$  of finite order, we see that  $\phi$  is an automorphism of order 2. However, it is shown in [1] that no manifold homotopy equivalent to  $M$  has an involution inducing the same outer automorphism on the fundamental group. We get a contradiction.



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